Explore Interesting Properties of AR(1) *

Letian Yu

November 2025

1 Introduction

Let $\{w_t : t = 0, 1, ...\}$ be a white noise process with variance σ_w^2 and let $|\phi| < 1$ be a constant. Consider the process $x_0 = w_0$ and

$$x_t = \phi x_{t-1} + w_t, t = 1, 2, ...$$

which simulates an AR(1) process from simulated white noise

2 Comment

First $|\phi| < 1$ make sure the process x_t is causal. But is it also a sufficient condition to being stationary? The answer is no.

We show that

$$x_t = \sum_{j=0}^{t} \phi_j w_{t-j}$$

Proof:

$$\begin{aligned}
x_t &= \frac{w_t}{1 - \phi B} && (1)\\
&= w_t(1 + \phi B + \phi^2 B^2 + \cdots) \qquad \text{Taylor's expansion} && (2)\\
&= w_t \sum_{j=0}^{\infty} (\phi B)^j && (3)\\
&= w_t \sum_{j=0}^{t} (\phi B)^j \qquad w_i = 0 \text{ with } i < 0 && (4)\\
&= \sum_{j=0}^{t} \phi^j w_{t-j} && (5)
\end{aligned}$$

*This article is inspired by Problem 3.2 in textbook TSA4

Thus

$$\mathbf{E}[x_t] = 0$$

And

$$\begin{aligned}
\mathrm{Var}(x_t) &= \mathrm{Var}\left(\sum_{j=0}^{t} \phi^j w_{t-j}\right) && (6)\\
&= \sum_{j=0}^{t} \mathrm{Var}(\phi^j w_{t-j}) \qquad \text{Variance sum law and } \mathrm{Cov}(w_i, w_j) = 0 && (7)\\
&= \sum_{j=0}^{t} \sigma_w^2 \phi^{2j} && (8)\\
&= \frac{(1-\phi^{2(t+1)})\sigma_w^2}{1-\phi^2} && (9)
\end{aligned}$$

Also for $h \geq 0$,

$$\begin{aligned}
\mathrm{Cov}(x_t, x_{t+h}) &= \mathbf{E}\left(\sum_{j=0}^{t+h} \phi^j w_{t+h-j} \sum_{j=0}^{t} \phi^j w_{t-j}\right) && (10)\\
&= \mathbf{E}\left(\sum_{i=0}^{t} \phi^i \phi^{i+h} \sigma_w^2\right) && (11)\\
&= \phi^h \mathrm{Var}(x_t) && (12)
\end{aligned}$$

Note that x_t is not stationary as its variance vary w.r.t t.

However, as $t \to \infty$, $\mathrm{Var}(x_t) \to \frac{\sigma_w^2}{1-\phi^2}$ is a constant, we say x_t is *asymptotically stationary*.

Consider the special case where $x_0 = \frac{w_0}{\sqrt{1-\phi^2}}$, then

$$\mathrm{Var}(x_0) = \frac{\sigma_w^2}{1-\phi^2} = \lim_{t\to\infty} \mathrm{Var}(x_t)$$

Therefore x_t will be stationary in this special case.

Indeed, if we compute the variance directly, *Proof.*

$$\operatorname{Var}(x_t) = \operatorname{Var}\left(\sum_{j=0}^{t-1} \phi_j w_{t-j} + \phi^t \frac{w_0}{\sqrt{1-\phi^2}}\right) \tag{13}$$

$$= \sigma_w^2 \sum_{j=0}^{t-1} (\phi^2)^j + \phi^{2t} \frac{\sigma_w^2}{1-\phi^2} \tag{14}$$

$$= \sigma_w^2 \left(\frac{1-\phi^{2t}+\phi^2 t}{1-\phi^2}\right) \tag{15}$$

$$= \frac{\sigma_w^2}{1-\phi^2} \tag{16}$$

□